NOTES TOWARD THE CONSTRUCTION OF NONLINEAR RELATIVISTIC QUANTUM FIELDS, I. THE HAMILTONIAN IN TWO SPACE-TIME DIMENSIONS AS THE GENERATOR OF A C-AUTOMORPHISM GROUP*

BY IRVING SEGAL

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Communicated by Eugene P. Wigner, March 20, 1967

1. One of the main foundational problems of quantum field theory dates back to Dirac's development[1] of radiation theory from the perturbation of a "free" Hamiltonian H_0 by an "interaction" Hamiltonian H_i. From a Hilbert space outlook which became applicable later, the situation was essentially that the total Hamiltonian $H = H_0 + H_i$ could be given mathematical existence as a bilinear form, in the sense that "first-order transition matrix" elements $\langle H\psi_1, \psi_2\rangle$ were well-defined for suitable state vectors ψ_1 and ψ_2 forming an extensive class within the state vector Hilbert spaces in question. However, as an operator in Hilbert space, H did not appear to have in its domain any nonzero vector; this circumstance left the foundations ambiguous, and was associated with profound difficulties in the treatment of second and higher-order approximations to the transition matrix elements. Although quantum field theory has been extensively developed since the time of Dirac, on the basis both of heuristic physical considerations and rigorous mathematical ones, the indicated unsatisfactory situation regarding the total field Hamiltonian remains unaltered to this day, from a foundational standpoint, for any nontrivial radiation theory.[2]

In a series of papers mainly in and before the 1950's, I essayed an approach in which the Hamiltonian does not appear as a self-adjoint operator in an *ad hoc* state vector space (such as the presumptive free quantum field state vector space), but rather as the generator of a group of automorphisms[3] of a C^*-algebra of field observables. In more conventional physical language, this meant that $e^{itH}Xe^{-itH} = X(t)$ might have an unexceptionable mathematical interpretation for conceptually measurable field observables X, although H itself might have no existence as an operator on the free quantum field Hilbert space $\mathbf{K}$, and $X(t)$ would be quite meaningless for an arbitrary bounded operator X on $\mathbf{K}$. (Explicit examples of such phenomena could readily be given, e.g., the quantized field associated with the partial differential equation $\Box\phi = m^2\phi + V(\mathbf{x})\phi$ when $V(\mathbf{x})$ is bounded and regular but not rapidly decreasing at infinity.) The conceptually measurable operators X in question are naturally taken, for theoretical reasons,[4] to form a C^*-algebra; in particular, the class of operators X for which $X(t)$ is appropriately definable is automatically a C^*-algebra. The distinction between C^*-algebras and the more restricted rings of operators (or W^*-algebras) introduced by Murray and von Neumann is here essential; in the case, for example, of the algebra of all (bounded) operators, every continuous one-parameter group of automorphisms is induced by a self-adjoint operator, and one is back to the original problem (except for the elimination of any zero-point energy).

On the other hand, somewhat paradoxically, this C^*-algebra approach leads ultimately in suitable cases to a Hamiltonian which is a self-adjoint operator in a specific Hilbert space, and not merely the generator of a one-parameter group of abstract C^*-automorphisms; this Hilbert space $\mathbf{K}_i$ is, however, entirely *a postiori*, and only has no explicit relation to the *a priori* free quantum field state vector space $\mathbf{K}$.[5] More specifically, a *vacuum state* is, within the C^*-framework, a state (in the sense of an expectation value form) which is temporally stationary, and such that the induced one-parameter unitary group in the Hilbert space associated with the state[6] has a nonnegative generator; this generator represents the total field Hamiltonian, acting on the state vector space $\mathbf{K}_i$ of the interacting field.[7] The free and interacting constituents H_0 and H_i of the total formal Hamiltonian are not represented on $\mathbf{K}_i$ in a natural fashion, and there is no apparent means to represent H directly on $\mathbf{K}_i$ as a sum of constituents representing H_0 and H_i. (These points may be exemplified in detail through consideration of the quantum field cited earlier, in which case $\mathbf{K}_i$ and the total Hamiltonian H as represented on $\mathbf{K}_i$ may be determined in closed form.)

The nonlinear terms in the fundamental partial differential equations of the relativistic theory, in which the central difficulties originate, were given a reinterpretation in reference 8 in terms of the Wick product at a precise time. This product, previously utilized primarily for the facility it provided in computations and the physical interpretation of perturbative processes, possesses invariant and characteristic algebraic features which provide a basis for its foundational use. Combining this idea with earlier developments, this note outlines progress in treating two-dimensional relativistic nonlinear fields under the approach indicated above. Although without direct physical relevance, the treatment of two-dimensional cases appears to be a necessary preliminary to that of four-dimensional cases, and two-dimensional models have attracted increasing theoretical attention of late; it appears actually that the four-dimensional case, while in part materially more singular, follows in part along parallel lines.

2. For succinctness and easier access to what is presently essential, generality and secondary details are avoided in the summary of our results on Wick products as

THEOREM 1. *Let $\phi(x, t)$ denote the conventional neutral scalar field in two space-time dimensions, as a self-adjoint-operator-valued mapping $f \rightarrow \Phi_t(f) = \int \phi(x,t)f(x)dx$, where f is arbitrary in the space $\mathbf{D}$ of real infinitely differentiable functions of compact support on the line, on the (free) Hilbert space $\mathbf{K}$. Let $\Phi(f) = \Phi_0(f)$ and $\dot{\Phi}(f) = \int \dot{\phi}(x,0)f(x)\,dx$.*

(a) *There exist unique maps $\Phi^{(n)}$ from $\mathbf{D}$ to the self-adjoint operators in $\mathbf{K}$, linear relative to strong linear operations on unbounded operators (i.e., the closures of the usual ones), such that*

$$\Phi^{(1)}(f) = \Phi(f);$$

$\Phi^{(n)}(f)$ has in its domain the free vacuum v and

$$\langle \Phi^{(n)}(f)v,v\rangle = 0;$$

the following commutation relations are valid in their bounded (exponentiated) form:[9]

$$[\Phi^{(m)}(f),\Phi^{(n)}(g)] = 0, \qquad [\dot{\Phi}(f), \Phi^{(n)}(g)] = in\Phi^{(n-1)}(fg).$$

(b) *If G is any open set on the line and f is supported by G, $f \in \mathbf{D}$, then $exp[i\Phi^{(n)}(f)]$ is in the ring of operators generated by the $exp[i\Phi(g)]$ as g varies over the elements of $\mathbf{D}$ supported by G.*

(c) *Let* $\mathbf{H}_r$ *denote the real Hilbert space of all normalizable solutions of the Klein-Gordon equation whose first time derivative vanishes at* $t = 0$, *and let* $\mathbf{K}$ *be represented as* $L_2(\mathbf{H}_r,d)$ *where* d *denotes the isonormal probability distribution in* $\mathbf{H}_r$ *according to the duality transform defined in reference 10. The operators* $\Phi^{(n)}(f)$ *are then correspondingly represented as the operations of multiplications by functions which are in* $L_p(\mathbf{H}_r,d)$ *for all* $p < \infty$ $(f \in \mathbf{D}, n = 1, 2, \ldots)$.

The proof may be made largely within the framework of generalized stochastic process theory on abelian groups.[11] More specifically, $\Phi(f)$ may be represented as a random variable, with expectation values corresponding to the vacuum expectation values, and is then a normal distribution of mean 0 and covariance $E(\Phi(f)\Phi(g)) = \int \hat{f}(k)\bar{\hat{g}}(k)(m^2 + k^2)^{-1/2}dk$. The argument becomes more transparent and the results usefully more general if Φ is permitted to be a process on an arbitrary locally compact abelian group G (written additively), of the form $\phi(x) \sim \int_{G^*} \lambda(x)\, \hat{\phi}(\lambda)\, d\lambda$, where G^* denotes the character group of G, and $d\lambda$ is the element of Haar measure on G^*, such that: (a) $\bar{\hat{\phi}}(\lambda) = \hat{\phi}(-\lambda)$, (b) $\hat{\phi}(\lambda)$ and $\hat{\phi}(\lambda')$ are stochastically independent unless $\lambda' = \pm \lambda$, (c) $E(\hat{\phi}(\lambda)\bar{\hat{\phi}}(\lambda')) = c(\lambda)\delta(\lambda - \lambda')$, with $c(\lambda) \in L_p(G^*)$ for all $p > 1$.[12]

If f is the characteristic function of a compact set in G^*, then the limit as $f \to 1$ of the weak distribution, $h \to \int \Phi(g_x)^2\, h(x)dx$, where $\hat{g} = f$ and $g_x(y) = g(y - x)$, will in general fail to exist; however, if from $\int \Phi(g_x)^2 h(x)dx$ is subtracted its expectation value, the resulting weak distribution does converge in L_2 on the domain of all continuous functions of compact support, as may be established by direct estimation of the variances involved. The result is then the second-order (generalized Wick) product

$$\Phi^{(2)}(h) = \lim_{f \to 1} \int [\Phi(g_x)^2 - E(\Phi(g_x)^2]h(x)dx$$

Similar estimates involving products of arbitrary even order of the $\hat{\phi}(\lambda)$ show that $\Phi^{(2)}(h)$ is pth-power integrable for all $p < \infty$.

The higher products are combinatorially quite complicated, and more complex subtractions are involved, but a basically similar analysis is effective. The variance of $\Phi^{(n)}(h)$ is bounded in all cases in terms of the n-fold convolution of $c(\lambda)$ with itself, and this remains in L_p for all $p < \infty$ by the Hausdorff-Young theorem. Having thus constructed the weak distributions $\Phi^{(n)}(f)$ explicitly, it follows from general spectral theory that they are self-adjoint and mutually commutative, and it is readily verifiable that $\Phi^{(n)}(f)$ has the indicated localization, in terms of the $\Phi(g)$.

3. Theorem 1 applies directly to the consideration of the spatially cut-off Hamiltonian for a scalar relativistic quantum field in two space-time dimensions. If p is any real polynomial, say $p(s) = a_0 + a_1 s + \ldots + a_k s^k$, $\Phi^{[p]}(f)$ is naturally defined as the closure of $a_0(\int f)I + a_1\Phi^{(1)}(f) + \ldots + a_k\Phi^{(k)}(f)$, and the spatially cut-off interaction Hamiltonian associated with the relativistic equation

$$\Box \phi = m^2\phi + p'(\phi) \qquad (*)$$

may be naturally taken, in the first instance, as the self-adjoint operator $H_i(f) = \Phi^{[p]}(f)$.

THEOREM 2. *Let* H_0 *denoted the conventional free-field Hamiltonian associated with (the linear part of) equation* (*). *Then* $H_0 + H_i(f)$ *is densely defined and has a self-adjoint extension* $H(f)$.

The proof depends on the use of the representation cited in part (c) of Theorem 1. A conventional n-particle state vector ψ is represented by a functional which is a polynomial of degree n on $\mathbf{H}_r$, and hence in $L_p(\mathbf{H}_r,d)$ for all $p < \infty$. The operator $\Phi^{[p]}(f)$ is that of multiplication by a functional which is likewise in $L_p(\mathbf{H}_r,d)$ for all $p < \infty$. It follows now from Hölder's inequality that the product of the latter functional with ψ is again in L_p for all $p < \infty$, and in particular in L_2, signifying that ψ is in the domain of $\Phi^{[p]}(f) = H_i(f)$.

It is easily seen that if the single-particle constituents of the n-particle state ψ are in the domain of the single-particle (free) Hamiltonian, then ψ is in the domain of H_0. Such suitably regular n-particle state vectors span a dense set in $\mathbf{K}$, showing that $H_0 + H_i(f)$ is densely defined. Both H_0 and $H_i(f)$ are invariant under the conjugation on $\mathbf{K}$ represented by complex conjugation on $L_2(\mathbf{H}_r,d)$, and hence so also is their sum, which implies according to a well-known result that it admits a self-adjoint extension.

4. The well-known domain of dependence properties of hyperbolic partial differential equations provides an intuitive basis for the existence of a theorem formulating the heuristic idea that a quantum field $\phi(x,t)$ satisfying equation (*) (in some sense) should be a function (in a suitable sense) of the operators $\phi(y,0)$, as y ranges over the classical region of control for the point (x,t) at the time 0. This would in turn suggest that, locally, the Heisenberg field operators

$$e^{itH(f)}\phi(x,0)\, e^{-itH(f)}$$

should be independent of f, if f is chosen to have the value 1 on a sufficiently large space region. In order to make a precise formulation, as well as to facilitate the later treatment of the vacuum, it is convenient to introduce the C^*-algebra $\mathbf{A}$ of all bounded operators on $\mathbf{K}$ which may be approximated uniformly by bounded operators in the ring of operators generated by the $\exp[i\Phi(f)]$ and $\exp[i\dot{\Phi}(g)]$ as f and g range over the elements of $\mathbf{D}$ supported by some bounded set B; any such bounded operator will be said to have support B, and $\mathbf{A}$ will be called the *space-finite* Weyl algebra (cf. ref. 13).

THEOREM 3. *There exists a one-parameter group $\Gamma(t)$ of automorphisms of the space-finite Weyl algebra which is uniquely determined by the condition that*

$$\Gamma(t)(X) = \exp[itH(f)]X\exp[-itH(f)]$$

whenever X has support B, and $f(x) = 1$ on a set C containing $B + [-t,t]$, provided the $H(f)$ are unique.

The proof uses an extension of the (Lie) product formula for the exponential of the sum of two operators. This is applicable by virtue of the essential self-adjointness of $H_0 + H_i(f)$ which results from the uniqueness assumption on $H(f)$.[14] Employing the localization property for the Wick product (part (b), Theorem 1) and the hyperbolic propagation character of the free field in a sense indicated earlier (which is readily deduced from a corresponding property of the associated classical equation), it follows that the propagation generated by $H(f)$ has the anticipated domains of dependence and regions of influence, and the indicated conclusion follows.[15]

5. The basic methods employed above are applicable to a variety of types of fields, in addition to scalar ones. In fact, the trilinear boson-fermion interaction

in two dimensions is in some respects simpler than the present ones. The uniqueness question will (in all probability) require further assumptions on the interaction Hamiltonian.[16] The same is true of the existence and uniqueness of a vacuum. Later notes are planned to treat these matters.

Summary.—For a general class of nonlinear relativistic quantum fields in two space-time dimensions, if f denotes any real nonnegative infinitely differentiable function of compact support, then the expression $H_0 + fH_i$, where H_0 and H_i denote natural mathematical formulations of the conventional free and interaction Hamiltonians, represents in a natural fashion a densely defined operator in the free field Hilbert space, having a self-adjoint extension $H(f)$. If these extensions are unique, the Heisenberg field corresponding to $H(f)$ has a limit as $f \to 1$ representing mathematically the motion induced by the relativistic Hamiltonian.

[1] Dirac, P. A. M., "The quantum theory of the emission and absorption of radiation," *Proc. Roy. Soc. London*, **A114**, 243–265 (1927).

[2] It was recognized quite early how H_0 might appropriately be represented as a self-adjoint operator. More recently it was shown that H_i could be represented as a self-adjoint operator (in the case of periodic boundary conditions in space) in a suitable representation of the field operators (Segal, I., *Ann. Math.*, **72**, 594–602 (1960)). However, the existence of a single representation in which both H_0 and H_i are represented by self-adjoint operators has remained open, as have, *a fortiori*, such questions as the existence of a nontrivial common domain, etc.

[3] An *automorphism* of an algebra of operators is here defined as a mapping $A \to \Gamma(A)$ which preserves all algebraic operations:

$$\Gamma(A + B) = \Gamma(A) + \Gamma(B),\ \Gamma(AB) = \Gamma(A)\Gamma(B),\ \Gamma(A^*) = (\Gamma(A))^*,\ \Gamma(cA) = c\Gamma(A).$$

[4] Cf. Segal, I., "Postulates for general quantum mechanics," *Ann. Math.*, **48**, 930–948 (1947).

[5] There is an analogy with a floating boundary value problem for a nonlinear partial differential equation.

[6] Cf. Segal, I., "A class of operator algebras which are determined by groups," *Duke Math. J.*, **18**, 221–265 (1951), Sec. 5.

[7] Essential uniqueness of the vaccum is anticipated, but is established as yet only for linear fields (Segal, I., *Ill. J. Math.*, **6**, 500–523 (1962)).

[8] Segal, I., *Compt. Rend.*, **259**, 301–303 (1964).

[9] These involve only the bounded operators $\exp[i\Phi^{(n)}(f)]$ and $\exp[i\dot{\phi}(g)]$, and may be given in closed form, but are more complicated than the infinitesimal commutation relations.

[10] Segal, I., "Tensor algebras over Hilbert spaces. I," *Trans. Am. Math. Soc.*, **18**, 106–134 (1956).

[11] Indeed, in the special case $m = 0$, the results are applicable to the Brownian motion process, and provide incidentally a definition for a pseudopower: $(x^{1/2}(t)^n$: of the fractional derivative of order $1/2$ of the Wiener process. The characterization of the Wick products in Theorem 1 is related to a characterization of the pseudopowers by their transformation properties under displacements in function space, whose effectiveness results from the absolute continuity and ergodicity of such displacements in Wiener space. Cf. Segal, I., in *Proceedings of the Conference on Functional Integration*, (M. I. T., April 1966), pp. 80–87.

[12] These conditions may be relaxed substantially, and Wick products relative to vacuum states other than the free vacuum thereby defined. Condition (*c*) fails in four-dimensional space time, since the function $c(\lambda)$ which intervenes is in L_p only for $p > 3$ and cannot be convolved with itself. The case in which $c(\lambda) \in L_1$ is that of a strict process (defined pointwise on G) and does not occur in a relativistic field theory.

[13] Segal, I., "Foundations of the theory of dynamical systems of infinitely many degrees of freedom," *Mat.-Fys. Medd. Danske Vid. Selsk.*, **31**, no. 12, 1–38 (1959).

[14] Cf. Nelson, E., "Feynman integrals and the Schrodinger equation," *J. Math. Phys.*, **5**, 332–343 (1964), Appendix B.

[15] Note that the total interaction Hamiltonian exists in the same sense, i.e., not necessarily as an operator but as an (unbounded) derivation of the Weyl algebra, or generator of a suitably continuous one-parameter group of automorphisms. Such a derivation may be identified with a self-adjoint operator in case the one-parameter group is induced by a one-parameter unitary group, whose self-adjoint generator is then uniquely determined (if required to annihilate the vacuum) and represents the derivation. Thus $H_0 + fH_t$ is in the first instance a derivation, but is here shown to represent in the foregoing sense an operator, under suitable conditions.

[16] Cf. the investigation of the case $\Box\phi = m^2\phi + g\phi^3$ with periodic boundary conditions by different methods by Nelson, E., "A quartic interaction in two dimensions," in *Mathematical Theory of Elementary Particles*, ed. R. W. Goodman and I. E. Segal (Cambridge: M. I. T. Press, 1966), pp. 69–74.